

On the self-adjointness and domain of Pauli-Fierz type Hamiltonians

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Abstract

We prove a general theorem about the self-adjointness and domain of Pauli-Fierz type Hamiltonians. Our proof is based on commutator arguments which allow us to treat fields with non-commuting components. As a corollary it follows that the domain of the Hamiltonian of non-relativistic QED with Coulomb interactions is independent of the coupling constant.

1 Introduction

Pauli-Fierz Hamiltonians are at the foundation of a mathematically consistent description of non-relativistic quantum mechanical matter interacting with the quantized electromagnetic field. For a Hamilton operator to describe a unitary dynamics it must be self-adjoint. Thus the question of self-adjointness is intimately related to physics. Knowing the domain of self-adjointness turns out to be of technical relevance for proving various properties about the Hamiltonian.

In this paper we prove a general theorem stating that the domains of Pauli-Fierz type Hamiltonians are independent of the coupling strength. Our proof is based on elementary commutator arguments, which allow us to treat fields of general form. Thus our theorem does not require the components of the fields to commute (see Theorem 6). In such a case functional integral methods are typically not applicable. As a corollary, we show

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that the domain of the Hamiltonian of non-relativistic QED with Coulomb interactions is independent of the coupling constant. Such a result has been obtained previously using functional integral methods, see [1, 2, 3, 4, 5]. However, an operator theoretic proof has so far been lacking in the literature.

The paper is organized as follows. First, we introduce definitions and collect some elementary properties in lemmas. Although these properties are well known, a proof is given in the Appendix for the convenience of the reader. The Hamiltonian of the interacting system is realized as the self-adjoint operator associated to a semi-bounded quadratic form. In a first step we show using a commutator argument that the domain of the free Hamiltonian is an operator core for the interacting Hamiltonian (see Lemma 11). In a second step we show using operator inequalities that the free Hamiltonian is operator bounded by the interacting Hamiltonian on a suitable core for the free Hamiltonian (see Lemma 12). Our result then follows as an application of the closed graph theorem.

2 Model and Statement of Result

Consider the Hilbert space $L^2(\mathbb{R}^n)$. For a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, we define the multiplication operator $M_f \varphi := f\varphi$ for all φ in the domain $D(M_f) = \{\varphi \in L^2(\mathbb{R}^n) | f\varphi \in L^2(\mathbb{R}^n)\}$. If f is real valued, then M_f is self-adjoint. Let p_j be the operator defined by, $p_j \varphi := -i\partial_j \varphi := -i(\partial_j \varphi)_{\text{dist}}$, for φ in the domain

$$D(p_j) := \{\psi \in L^2(\mathbb{R}^n) | (\partial_j \psi)_{\text{dist}} \in L^2(\mathbb{R}^n)\},$$

where $(\cdot)_{\text{dist}}$ stands for the distributional derivative and ∂_j stands for the partial derivative with respect to the j -th coordinate in \mathbb{R}^n . The Laplacian is defined by $-\Delta := p^2 := \sum_{j=1}^n p_j^2$ with domain $D(p^2) := H^2(\mathbb{R}^n)$. The operators p_j and p^2 are self-adjoint on their domains.

In this paragraph, we review some standard conventions about tensor products, which can be found for example in [6]. The algebraic tensor product $V \otimes W$ of the vector spaces V and W consists of all finite linear combinations of vectors of the form $\varphi \otimes \eta$ with $\varphi \in V$ and $\eta \in W$. For \mathcal{H} and \mathcal{K} two Hilbert spaces the tensor product of Hilbert spaces is the closure of the algebraic tensor product of \mathcal{H} and \mathcal{K} in the topology induced by the inner product. We adopt the standard convention that $V \otimes W$ denotes the tensor product of Hilbert spaces if V and W are Hilbert spaces; if V or W is a non complete inner product space then $V \otimes W$ denotes the algebraic tensor product. For A and B closed operators in the Hilbert spaces \mathcal{H} and \mathcal{K} , respectively, we denote by $A \otimes 1$ the closure of

$A \otimes 1 \upharpoonright D(A) \otimes \mathcal{K}$ and by $1 \otimes B$ the closure of $1 \otimes B \upharpoonright \mathcal{H} \otimes D(B)$. If A is essentially self-adjoint on D_a , then $A \otimes 1$ is essentially self-adjoint on $D_a \otimes \mathcal{K}$. An analogous statement holds for $1 \otimes B$. For notational convenience, the operators $A \otimes 1$ and $B \otimes 1$ are written as A and B , respectively. No confusion should arise, since it should be clear from the context in which space the operator acts. By associativity and bilinearity of the tensor product, the above definitions, conventions, and properties generalize in a straight forward way to multiple tensor products, [6].

Let \mathfrak{h} be a separable complex Hilbert space and let $\otimes^n \mathfrak{h} = \mathfrak{h} \otimes \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$ denote the n -fold tensor product of \mathfrak{h} with itself. We define the Hilbert spaces

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n, \quad \mathcal{F}_0 := \mathbb{C}, \quad \mathcal{F}_n := S_n(\otimes^n \mathfrak{h}), \quad n \geq 1,$$

where S_n denotes the orthogonal projection onto totally symmetric tensors, i.e., the projection satisfying $S_n(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \varphi_{\sigma(1)} \otimes \varphi_{\sigma(2)} \otimes \cdots \otimes \varphi_{\sigma(n)}$, with \mathcal{S}_n being the set of permutations of the numbers 1 through n . By definition, a vector $\psi \in \mathcal{F}$ is a sequence $(\psi_{(n)})_{n \geq 0}$ of vectors $\psi_{(n)} \in \mathcal{F}_n$ such that its norm $(\sum_{n=0}^{\infty} \|\psi_{(n)}\|^2)^{1/2}$ is finite. Let $\Omega = (1, 0, 0, \dots)$, and let

$$\mathcal{F}_{\text{fin}} = \{\psi \in \mathcal{F} \mid \psi_{(n)} = 0 \text{ except for finitely many } n\}$$

denote the subspace consisting of states containing only finitely many “particles”. Let A be a self-adjoint operator on \mathfrak{h} with domain $D(A)$. The second quantization $d\Gamma(A)$ is an operator in \mathcal{F} defined as follows. Let $A_{(n)}$ be the closure of

$$(A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A) \upharpoonright S_n(\otimes^n D(A)).$$

Define $(d\Gamma(A)\psi)_{(n)} = A_{(n)}\psi_{(n)}$ for all ψ in the domain $D(d\Gamma(A)) := \{\psi \in \mathcal{F} \mid \psi_{(n)} \in D(A_{(n)}), \sum_{n=0}^{\infty} \|A_{(n)}\psi_{(n)}\|^2 < \infty\}$. It follows from the definition that $d\Gamma(A)$ is self-adjoint. The number operator is defined by $N = d\Gamma(1)$. For each $h \in \mathfrak{h}$ we define the creation operator $a^*(h)$ by

$$a^*(h)\varphi = (n+1)^{1/2} S_{n+1} h \otimes \varphi, \quad \forall \varphi \in \mathcal{F}_n,$$

and extend $a^*(h)$ to be an operator in \mathcal{F} by taking the closure. Let $a(h)$ be the adjoint of $a^*(h)$. The annihilation operator $a(h)$ acts on \mathcal{F}_0 as the zero operator and on vectors $S_n(\varphi_1 \otimes \cdots \otimes \varphi_n) \in \mathcal{F}_n$, with $n \geq 1$, as

$$a(h)S_n(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n) = n^{-1/2} S_{n-1} \sum_{i=1}^n (h, \varphi_i) \varphi_1 \otimes \cdots \otimes \varphi_{i-1} \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_n.$$

For $h \in \mathfrak{h}$, we introduce the field operator on \mathcal{F}_{fin}

$$\widehat{\phi}(h) = 2^{-1/2}(a(h) + a^*(h)) \ .$$

This operator is symmetric, and hence closable. Let $\phi(h)$ denote the closure of $\widehat{\phi}(h)$.

We shall henceforth assume that $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^p)$ and that $\omega : \mathbb{R}^n \rightarrow [0, \infty)$ is a measurable function which is a.e. nonzero. The field energy, defined by,

$$H_f = d\Gamma(M_\omega)$$

is self-adjoint in \mathcal{F} . It is notationally convenient to define the Hilbert space $\mathfrak{h}_\omega := \{h \in \mathfrak{h} \mid \|h\|_\omega < \infty\}$ with norm $\|h\|_\omega := (\|h\|^2 + \|h/\sqrt{\omega}\|^2)^{1/2}$. In the next lemma we collect some basic and well known properties. A proof of the lemma can be found in the Appendix.

Lemma 1. *The following statements hold.*

(a) For $g, h \in \mathfrak{h}$,

$$[\phi(g), \phi(h)] = i\text{Im}(g, h) \quad \text{on } \mathcal{F}_{\text{fin}} \ . \quad (1)$$

(b) If $h \in \mathfrak{h}_\omega$, then $D(H_f^{1/2}) \subset D(\phi(h))$ and

$$\|\phi(h)(H_f + 1)^{-1/2}\| \leq 2^{1/2}\|h\|_\omega \ . \quad (2)$$

If $g, h \in \mathfrak{h}_\omega$, then $D(H_f) \subset D(\phi(g)\phi(h))$ and

$$\|\phi(g)\phi(h)(H_f + 1)^{-1}\| \leq 4\|g\|_\omega\|h\|_\omega \ . \quad (3)$$

(c) If $h, \omega h \in \mathfrak{h}$, then

$$[H_f, \phi(h)] = -i\phi(i\omega h) \quad \text{on } \mathcal{F}_{\text{fin}} \cap D(H_f) \ . \quad (4)$$

Now we will extend the above definition to the tensor product $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathcal{F}$ of Hilbert spaces. We will use the natural isomorphism of Hilbert spaces,

$$\mathcal{H} := L^2(\mathbb{R}^n) \otimes \mathcal{F} \cong L^2(\mathbb{R}^n; \mathcal{F}) \ ,$$

and we introduce the space

$$\mathcal{H}_{\text{fin}} = \{\psi \in \mathcal{H} \mid \psi_{(n)} = 0 \ , \text{ except for finitely many } n\} \ .$$

Let $L^\infty(\mathbb{R}^d; \mathfrak{h})$ and $L^\infty(\mathbb{R}^d; \mathfrak{h}_\omega)$ denote the Banach spaces of measurable functions from \mathbb{R}^d to \mathfrak{h} and \mathfrak{h}_ω with norms $\|G\|_\infty := \text{ess sup}_{x \in \mathbb{R}^d} \|G(x)\|$ and $\|G\|_{\omega, \infty} := \text{ess sup}_{x \in \mathbb{R}^d} \|G(x)\|_\omega$, respectively. For $G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ define $\widehat{\Phi}(G)$ for $\psi \in \mathcal{H}_{\text{fin}}$ by

$$(\widehat{\Phi}(G)\psi)(x) = \phi(G(x))\psi(x) .$$

Note that $\widehat{\Phi}(G)$ is a symmetric operator and hence closable. Let $\Phi(G)$ denote the closure of $\widehat{\Phi}(G)$.

Remark. Although not needed for the proof of the theorem, we note that $\phi(f)$ and $\Phi(G)$ are essentially self-adjoint on \mathcal{F}_{fin} and \mathcal{H}_{fin} , respectively. This can be shown using, for example, Nelson's analytic vector theorem, see [7].

Lemma 2. *Let $G \in L^\infty(\mathbb{R}^d; \mathfrak{h}_\omega)$. Then $D(H_f^{1/2}) \subset D(\Phi(G))$ and*

$$\|\Phi(G)(H_f + 1)^{-1/2}\| \leq 2^{1/2}\|G\|_{\omega, \infty} \quad (5)$$

Proof. Follows from inequality (2). □

Lemma 3. *Let $(G_i)_{i=1}^n \subset L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$, and $A_j := \Phi(G_j)$. Then the quadratic form*

$$q(\varphi, \psi) := \sum_{j=1}^n ((p_j + A_j)\varphi, (p_j + A_j)\psi) + (H_f^{1/2}\varphi, H_f^{1/2}\psi) , \quad (6)$$

defined on the form domain $Q(q) := \bigcap_i D(p_i) \cap D(H_f^{1/2})$ is nonnegative and closed.

The proof of this Lemma is given in the Appendix.

Definition 4. *For $(G_j)_{j=1}^n \subset L^2(\mathbb{R}^n; \mathfrak{h}_\omega)$, let T_A be the unique self-adjoint operator associated to the quadratic form (6). For $G_j \in L^2(\mathbb{R}^n; \mathfrak{h})$, we will set $A_j := \Phi(G_j)$.*

Remark. By the first representation theorem for quadratic forms, T_A is characterized as follows:

$$(T_A\varphi, \psi) = q(\varphi, \psi) \quad , \quad \forall \psi \in Q(q) , \quad (7)$$

for all φ in the domain $D(T_A) = \{\varphi \in Q(q) | \exists \eta \in \mathcal{H}, \forall \psi \in C, q(\varphi, \psi) = (\eta, \psi)\}$, where C is any form core for q .

Definition 5. $G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ is said to be weakly ∂_j -differentiable if there is a $K \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ such that for all $v \in \mathfrak{h}$ and all $f \in C_0^\infty(\mathbb{R}^n)$

$$\int \partial_j f(x)(v, G(x)) dx = - \int f(x)(v, K(x)) dx ;$$

in that case we write $\partial_j G = K$.

Hypothesis (G). $(G_j)_{j=1}^n \subset L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$ is a collection of functions such that G_j is weakly ∂_j -differentiable and $\omega G_j, \sum_{l=1}^n \partial_l G_l \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$.

We will adopt standard conventions for the sum and the composition of two operators: $D(S + R) = D(S) \cap D(R)$ and $D(SR) = \{\psi \in D(R) | R\psi \in D(S)\}$. Since p^2 and H_f are commuting positive operators, $p^2 + H_f$ is self-adjoint on the domain $D(p^2 + H_f) = D(p^2) \cap D(H_f)$.

Theorem 6. Let Hypothesis (G) hold. Then T_A is essentially self-adjoint on any operator core for $p^2 + H_f$ and $D(T_A) = D(p^2 + H_f)$.

The next theorem relates T_A with a natural definition. By $(p + A)^2$ we denote the operator sum $\sum_j (p_j + A_j)^2$. Thus by definition

$$\begin{aligned} \varphi \in D((p + A)^2) \\ \iff \varphi \in D(p_j) \cap D(A_j) \text{ and } (p_j \varphi + A_j \varphi) \in D(p_j) \cap D(A_j) , \quad \forall j = 1, \dots, n ; \end{aligned}$$

and $D((p + A)^2 + H_f) = D((p + A)^2) \cap D(H_f)$.

Theorem 7. Suppose $G_j \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$ is weakly ∂_j -differentiable and $\partial_j G_j \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$ for $j = 1, \dots, n$. Then $D(p^2 + H_f) \subset D(T_A) \cap D((p + A)^2 + H_f)$. Furthermore,

$$(p + A)^2 \varphi + H_f \varphi = T_A \varphi , \quad \text{if } \varphi \in D(p^2 + H_f) .$$

3 Applications

Let $\mathcal{H} = \otimes^n (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2) \cong L^2(\mathbb{R}^{3n}; \otimes^n \mathbb{C}^2)$ be the Hilbert space, describing n spin- $\frac{1}{2}$ particles. Let x_j denote the coordinate of the j -th particle having mass $m_j > 0$, and let $x = (x_1, \dots, x_n) \in \mathbb{R}^{3n}$. Let

$$\sigma_{j,a} = 1 \otimes \dots \otimes 1 \otimes \sigma_a \otimes 1 \dots \otimes 1 ,$$

where σ_a , the a -th Pauli matrix, acts on the j -th factor of $\otimes^n \mathbb{C}^2$. Let $\mathfrak{h} = L^2(\mathbb{R}^3; \mathbb{C}^2)$, and let $\varepsilon(1, k)$ and $\varepsilon(2, k)$ be normalized vectors in \mathbb{C}^3 depending measurably on $k/|k|$ such that $(\varepsilon(i, k), k) = 0$, for $i = 1, 2$ and $(\varepsilon(1, k), \varepsilon(2, k)) = 0$. Let $\omega(k) = \sqrt{m_{\text{ph}}^2 + k^2}$ for some $m_{\text{ph}} \geq 0$. Let $\rho(k)$ be a function such that $\rho/\omega, \sqrt{\omega}\rho \in L^2(\mathbb{R}^3)$. For $a = 1, 2, 3$ and $j = 1, \dots, n$, let

$$[G_{j,a}(x)](k, \lambda) = \frac{\rho(k)}{\sqrt{\omega(k)}} e^{-ik \cdot x_j} \varepsilon_a(\lambda, k), \quad [E_{j,a}(x)](k, \lambda) = \frac{-i\rho(k)}{\sqrt{\omega(k)}} e^{-ik \cdot x_j} (k \wedge \varepsilon(\lambda, k))_a, \quad (8)$$

and $A_{j,a} = \Phi(G_{j,a})$ and $B_{j,a} = \Phi(E_{j,a})$. Let $V_c : \mathbb{R}^{3n} \rightarrow \mathbb{R}$ be a function which is infinitesimally bounded with respect to $-\Delta := p^2$. For example this is the case, if for $c_{j,l}, z_{j,J} \in \mathbb{R}$ and $(R_J)_{J=1}^M \subset \mathbb{R}^3$,

$$V_c = \sum_{j \neq l} \frac{c_{j,l}}{|x_j - x_l|} + \sum_{j=1}^n \sum_{J=1}^M \frac{z_{j,J}}{|x_j - R_J|}.$$

We want to point out that one usually imposes the constraint that $\rho(k) = \overline{\rho(-k)}$, which is not needed for the corollary below to hold. Moreover, note that $[A_{a,l}, A_{b,j}] = 0$ is satisfied only if $|\rho(k)| = |\rho(-k)|$ (see Lemma 9).

Corollary 8. *The operator*

$$\sum_j \frac{1}{2m_j} (-i\nabla_j - e_j A_j)^2 + H_f + \sum_j \frac{e_j}{2m_j} \boldsymbol{\sigma}_j \cdot B_j + V_c, \quad (9)$$

with $e_j \in \mathbb{R}$, is well defined on $D(\sum_j \frac{-\Delta_j}{2m_j} + H_f)$. It is self-adjoint with this domain, essentially self-adjoint on any core for $\sum_j \frac{-\Delta_j}{2m_j} + H_f$, and bounded from below.

Clearly the same result holds if we restrict the operators to subspaces taking into account certain particle statistics. The statement of this corollary has been previously obtained using functional integral methods, [5].

Proof. After rescaling the particle coordinates and the functions (8), we can assume that $m_j = 1$ and $e_j = -1$. The $G_{j,a}$ (possibly a rescaled version thereof) satisfy the assumptions of Theorems 6 and 7. Thus by Theorem 7, $(p + A)^2 + H_f$ is well defined on $D(p^2 + H_f)$. Moreover, for $\varphi \in D(p^2 + H_f)$, we have $(p + A)^2 \varphi + H_f \varphi = T_A \varphi$. By Theorem 6, $D(p^2 + H_f) = D(T_A)$ and therefore T_A is $p^2 + H_f$ bounded. Since $\boldsymbol{\sigma}_j \cdot B_j$ and V_c are infinitesimally small with respect to $p^2 + H_f$, the claim follows now from Kato's Theorem. \square

4 Proofs

We use the convention that $[S, R]$ stands for the operator $SR - RS$ defined on the domain $D([S, R]) = \{\psi \in D(R) \cap D(S) \mid S\psi \in D(R), R\psi \in D(S)\}$.

Lemma 9. *The following statements hold.*

(a) For $F, G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$,

$$[\Phi(F), \Phi(G)] = i\text{Im}(F, G)_{\mathfrak{h}} \quad \text{on } \mathcal{H}_{\text{fin}},$$

where the right hand side is a multiplication operator and the inner product is taken in \mathfrak{h} .

(b) If $F, G \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$, then $D(H_f^{1/2}) \subset D(\Phi(G))$ and

$$\|\Phi(G)(H_f + 1)^{-1/2}\| \leq 2^{1/2}\|G\|_{\omega, \infty}$$

moreover, $D(H_f) \subset D(\Phi(F)\Phi(G))$ and

$$\|\Phi(F)\Phi(G)(H_f + 1)^{-1}\| \leq 4\|F\|_{\omega, \infty}\|G\|_{\omega, \infty}.$$

(c) For $G, \omega G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$,

$$[H_f, \Phi(G)] = -i\Phi(i\omega G) \quad \text{on } \mathcal{H}_{\text{fin}} \cap D(H_f).$$

(d) Let $G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ be weakly ∂_j -differentiable. Then $\Phi(G)$ leaves $\mathcal{H}_{\text{fin}} \cap D(p_j)$ invariant and

$$[p_j, \Phi(G)] = -i\Phi(\partial_j G), \quad \text{on } \mathcal{H}_{\text{fin}} \cap D(p_j).$$

Proof. All statements up to and including (c) follow directly from the definition and corresponding statements in Lemma 1. (d) Follows from Lemma 13 in the Appendix. \square

In the proof, we will use certain commutator identities which can be easily verified on a suitable core, which we shall now introduce. Let

$$\mathcal{C} := C_0^\infty(\mathbb{R}^n) \otimes \left(\prod_{n=0}^{\infty} S_n(\otimes^n C^\omega) \right),$$

where C^ω denotes the set of functions f in $L^2(\mathbb{R}^d; \mathbb{C}^p)$ with $\text{supp } f \subset \bigcup_{m=0}^{\infty} \{k \mid \omega(k) \leq m\}$, and $\prod_{n=0}^{\infty} S_n(\otimes^n C^\omega)$ denotes the set of all sequences $(\psi_{(n)})_{n=0}^{\infty}$ such that $\psi_{(n)} \in S_n(\otimes^n C^\omega)$

and $\psi_{(n)} = 0$ for all but finitely many n . Note that $\mathcal{C} \subset \bigcap_{m=1}^{\infty} D(H_f^m)$. If $G_j \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ is weakly ∂_j -differentiable, we have by Lemma 9 (d),

$$\mathcal{C} \subset D(p_j^2) \cap D(A_j p_j) \cap D(p_j A_j) \cap D(A_j^2). \quad (10)$$

Lemma 10. *Let $(G_j)_{j=1}^n \subset L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$. The set \mathcal{C} is a form core for q .*

Proof. By definition we have to show that \mathcal{C} is dense in $(Q(q), \|\cdot\|_+)$, where

$$\|\varphi\|_{+1}^2 := \|\varphi\|^2 + \sum_j \|(p_j + A_j)\varphi\|^2 + \|H_f^{1/2}\varphi\|^2. \quad (11)$$

For $\psi \in Q(q)$, there exists a sequence $(\psi_n)_{n=0}^\infty \subset \mathcal{C}$ such that $\psi_n \rightarrow \psi$, $p_j \psi_n \rightarrow p_j \psi$, and $H_f^{1/2} \psi_n \rightarrow H_f^{1/2} \psi$. This and Lemma 9 (b) imply that $(p_j + A_j)\psi_n \rightarrow (p_j + A_j)\psi$. Thus \mathcal{C} is dense in $(Q(q), \|\cdot\|_+)$. \square

Part (c) of the next lemma immediately implies Theorem 7. Parts (a),(b) and (d) will be used to prove Theorem 6.

Lemma 11. *Suppose for $j = 1, \dots, n$, $G_j \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ is weakly ∂_j -differentiable. Then following statements are true.*

- (a) *For all $\varphi \in \mathcal{C}$, $T_A \varphi = (p + A)^2 \varphi + H_f \varphi$.*
- (b) *Let $G_j, \sum_l \partial_l G_l \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$ for all $j = 1, \dots, n$. Then $D(p^2 + H_f) \subset D(T_A)$.*
- (c) *Let $G_j, \partial_j G_j \in L^\infty(\mathbb{R}^n; \mathfrak{h}_\omega)$ for all $j = 1, \dots, n$. Then $D(p^2 + H_f) \subset D((p + A)^2) \cap D(T_A)$ and for all $\varphi \in D(p^2 + H_f)$, $T_A \varphi = (p + A)^2 \varphi + H_f \varphi$.*
- (d) *If Hypothesis (G) holds, the set $D(p^2 + H_f)$ is an operator core for T_A .*

Proof. (a). Using (10), we see that for $\varphi, \psi \in \mathcal{C}$, $q(\varphi, \psi) = \sum_j ((p + A)_j^2 \varphi, \psi) + (H_f \varphi, \psi)$. This shows (a).

(b). Let $\varphi \in D(p^2 + H_f)$. By definition $\varphi \in D(p^2) \cap D(H_f)$. By Lemma 9, $\varphi \in D(A_l^2)$. Since $\|H_f^{1/2} p_l \varphi\|^2 \leq \|H_f \varphi\|^2 + \|p_l^2 \varphi\|^2$, we have $p_l \varphi \in D(A_l)$. Thus it follows that for $\psi \in \mathcal{C}$,

$$q(\varphi, \psi) = \sum_j \{ (p_j^2 \varphi, \psi) + (A_j p_j \varphi, \psi) + (A_j^2 \varphi, \psi) + (\varphi, A_j p_j \psi) \} + (H_f \varphi, \psi). \quad (12)$$

Now using Lemma 9 (d), we see that the summation over the last term in the sum yields,

$$\begin{aligned}\sum_j (\varphi, A_j p_j \psi) &= (\varphi, i\Phi(\Sigma_j \partial_j G_j) \psi) + \sum_j (\varphi, p_j A_j \psi) \\ &= (-i\Phi(\Sigma_j \partial_j G_j) \varphi, \psi) + \sum_j (A_j p_j \varphi, \psi) .\end{aligned}$$

Thus, there exists an $\eta \in \mathcal{H}$, such that for all $\psi \in \mathcal{C}$, $q(\varphi, \psi) = (\eta, \psi)$. This shows (b).

(c). In view of (b) and (12), we only need to show $A_j \varphi \in D(p_j)$. By Lemma 9 (d), for $\varphi_n = \chi_{[0,n]}(N)\varphi$ with $\chi_{[0,n]}$ denoting the characteristic function of the set $[0, n]$,

$$p_j A_j \varphi_n = -i\phi(\partial_j G_j) \varphi_n + A_j p_j \varphi_n .$$

Since the limit of the right hand side exists and $A_j \varphi_n$ converges, it follows that $A_j \varphi \in D(p_j)$.

(d). Let $\alpha > 0$. For notational compactness, we set $R_\alpha := (\alpha H_f + 1)^{-1}$ and $\Pi_j := \Phi(i\omega G_j)$. Moreover, observe that $D(|p|) = \cap_j D(p_j)$.

Step 1: For $\varphi \in D(T_A)$, and for all $\psi \in \mathcal{C}$,

$$q(R_\alpha \varphi, \psi) = q(\varphi, R_\alpha \psi) + (F_\alpha \varphi, \psi) + 2 \sum_j (E_{\alpha,j} (p_j + A_j) \varphi, \psi) , \quad (13)$$

where $E_{\alpha,j}$ and F_α are bounded operators defined by

$$\begin{aligned}E_{\alpha,j} &= -iR_\alpha \alpha \Pi_j R_\alpha , \\ F_\alpha &= - \sum_j 2\alpha^2 R_\alpha (\Pi_j R_\alpha)^2 - \alpha R_\alpha^2 \sum_j (\omega G_j, G_j)_{\mathfrak{h}} + i[R_\alpha, \Phi(\Sigma_j \partial_j G_j)] .\end{aligned}$$

To show this, let $\varphi \in D(T_A)$ and $\psi \in \mathcal{C}$. By definition $\varphi \in D(|p|)$. Since R_α is bounded and acts on a different factor of the tensor product it leaves $D(|p|)$ invariant. It follows that $R_\alpha \varphi \in Q(q)$. By the definition of the quadratic form,

$$q(R_\alpha \varphi, \psi) = \sum_j (\varphi, R_\alpha (p_j + A_j)^2 \psi) + (H_f^{1/2} \varphi, H_f^{1/2} R_\alpha \psi) . \quad (14)$$

We write the summand in the first expression on the right as

$$\begin{aligned}(\varphi, R_\alpha (p_j + A_j)^2 \psi) &= (\varphi, [R_\alpha, (p_j + A_j)^2] \psi) + (\varphi, (p_j + A_j)^2 R_\alpha \psi) \\ &= (\varphi, [R_\alpha, (p_j + A_j)^2] \psi) + ((p_j + A_j) \varphi, (p_j + A_j) R_\alpha \psi)\end{aligned}$$

Inserting this into (14) we find

$$q(R_\alpha \varphi, \psi) = q(\varphi, R_\alpha \psi) + \sum_j (\varphi, [R_\alpha, (p_j + A_j)^2] \psi) .$$

We calculate the commutator

$$\begin{aligned} & \sum_j (\varphi, [R_\alpha, (p_j + A_j)^2] \psi) \\ &= \sum_j \{ (\varphi, [[R_\alpha, A_j], p_j + A_j] \psi) + (\varphi, 2(p_j + A_j)[R_\alpha, A_j] \psi) \} \\ &= \sum_j \{ (\varphi, [[R_\alpha, A_j], A_j] \psi) + ((\varphi, [R_\alpha, [A_j, p_j]] \psi) - (2(p_j + A_j)\varphi, E_{\alpha,j} \psi)) \} \\ &= (F_\alpha \varphi, \psi) + 2 \sum_j (E_{\alpha,j} (p_j + A_j) \varphi, \psi) , \end{aligned}$$

where we used that on \mathcal{H}_{fin} ,

$$E_{\alpha,j} = [A_j, R_\alpha] \quad , \quad [[R_\alpha, A_j], A_j] = -2\alpha^2 R_\alpha (\Pi_j R_\alpha)^2 - \alpha R_\alpha^2 (\omega G_j, G_j)_\mathfrak{h} .$$

Step 2: For all $\varphi \in D(T_A)$, $R_\alpha \varphi \in D(T_A)$ and $\lim_{\alpha \downarrow 0} T_A R_\alpha \varphi = T_A \varphi$.

From Eq. (13), it follows that $R_\alpha \varphi \in D(T_A)$ and that

$$T_A R_\alpha \varphi = R_\alpha T_A \varphi + F_\alpha \varphi + 2 \sum_j E_{\alpha,j} (p_j + A_j) \varphi . \quad (15)$$

By the spectral theorem $s - \lim_{\alpha \downarrow 0} R_\alpha = 1$. Using the estimate

$$\|\alpha^{1/2} \Pi_j (\alpha H_f + 1)^{-1/2}\| \leq \max(1, \alpha^{1/2}) \|\Pi_j (H_f + 1)^{-1/2}\| ,$$

we see that $E_{\alpha,j}$ and the first term of F_α converge to 0 for $\alpha \downarrow 0$. Moreover,

$$\Phi(\Sigma_j \partial_j G_j) R_\alpha \varphi = \Phi(\Sigma_j \partial_j G_j) (H_f + 1)^{-1/2} R_\alpha (H_f + 1)^{1/2} \varphi \rightarrow \Phi(\Sigma_j \partial_j G_j) \varphi ,$$

as $\alpha \downarrow 0$, which implies $\lim_{\alpha \downarrow 0} [R_\alpha, \Phi(\Sigma_j \partial_j G_j)] \varphi = 0$. Thus the right hand side of Eq. (15) converges for $\alpha \downarrow 0$ to $T_A \varphi$.

Step 3: For $\varphi \in D(T_A)$, and $\alpha > 0$, $R_\alpha \varphi \in D(p^2) \cap D(H_f)$.

It is clear that $R_\alpha \varphi \in D(H_f)$. Let $\varphi \in D(T_A)$. Then by (13) there exists an $\eta \in \mathcal{H}$, such that for all $\psi \in \mathcal{C}$,

$$\begin{aligned} (\eta, \psi) &= \sum_j ((p_j + A_j) R_\alpha \varphi, (p_j + A_j) \psi) \\ &= \sum_j \{ (p_j R_\alpha \varphi, p_j \psi) + (A_j^2 R_\alpha \varphi, \psi) + (p_j R_\alpha \varphi, A_j \psi) + (A_j R_\alpha \varphi, p_j \psi) \} \end{aligned}$$

Furthermore, using $\sum_j (A_j R_\alpha \varphi, p_j \psi) = (R_\alpha \varphi, i\Phi(\sum_j \partial_j G_j) \psi) + \sum_j (R_\alpha \varphi, p_j A_j \psi)$, $\varphi \in D(|p|)$, and Lemma 9, we see that there exists an $\eta_1 \in \mathcal{H}$, such that

$$\sum_j (p_j R_\alpha \varphi, p_j \psi) = (\eta_1, \psi) \quad , \quad \forall \psi \in \mathcal{C} .$$

This implies $R_\alpha \varphi \in D(p^2)$, since \mathcal{C} is a form core for p^2 . \square

Lemma 12. *Let Hypothesis (G) hold. Then there exists constants C_1, C_2 such that for all $\varphi \in \mathcal{C}$,*

$$\|(p^2 + H_f)\varphi\|^2 \leq C_1 \|((p + A)^2 + H_f)\varphi\|^2 + C_2 \|\varphi\|^2 . \quad (16)$$

Proof. The proof will be based on the relations given in Lemma 9. First observe that

$$\|(p^2 + H_f)\varphi\|^2 \leq 2\|p^2 \varphi\|^2 + 2\|H_f \varphi\|^2 , \quad \forall \varphi \in \mathcal{C} . \quad (17)$$

The lemma will follow as a direct consequence of Inequality (17) and Steps 1 and 2, below.

Step 1: There exist constants c_1, c_2, c_3 such that

$$\|p^2 \varphi\|^2 \leq c_1 \|((p + A)^2 \varphi)\|^2 + c_2 \|H_f \varphi\|^2 + c_3 \|\varphi\|^2 , \quad \forall \varphi \in \mathcal{C} .$$

We have

$$\begin{aligned} \|p^2 \varphi\|^2 &= \|((p + A)^2 - A \cdot (p + A) - (p + A) \cdot A + A^2) \varphi\|^2 \\ &\leq 3\|(p + A)^2 \varphi\|^2 + 3\|(A \cdot (p + A) + (p + A) \cdot A) \varphi\|^2 + 3\|A^2 \varphi\|^2 , \end{aligned}$$

writing $A^2 = A \cdot A$. We estimate the middle term using the notation $[p, A] := \sum_j [p_j, A_j]$,

$$\begin{aligned} \|(A \cdot (p + A) + (p + A) \cdot A) \varphi\|^2 &= \|(2A \cdot (p + A) + [p, A]) \varphi\|^2 \\ &\leq 8\|A \cdot (p + A) \varphi\|^2 + 2\|[p, A] \varphi\|^2 . \end{aligned} \quad (18)$$

The second term on the last line is estimated using $\|[p, A] \varphi\|^2 \leq C\|(H_f + 1)^{1/2} \varphi\|^2$. Here and below C denotes a constant which may change from one inequality to the next. The first term in (18) is estimated as follows:

$$\|A \cdot (p + A) \varphi\|^2 \leq C \sum_j \|A_j (p_j + A_j) \varphi\|^2 \leq C \sum_j \|(H_f + 1)^{1/2} (p_j + A_j) \varphi\|^2 .$$

Further, using a commutator

$$\begin{aligned}
\sum_j \|(H_f + 1)^{1/2}(p_j + A_j)\varphi\|^2 &= \sum_j ((p_j + A_j)\varphi, (H_f + 1)(p_j + A_j)\varphi) \\
&= \sum_j (((p_j + A_j)^2\varphi, (H_f + 1)\varphi) + ((p_j + A_j)\varphi, [H_f, A_j]\varphi)) \\
&\leq C(\|(p + A)^2\varphi\|^2 + \sum_j \|(p_j + A_j)\varphi\|^2 + \|(H_f + 1)\varphi\|^2) \\
&\leq C(\|(p + A)^2\varphi\|^2 + \|(H_f + 1)\varphi\|^2) .
\end{aligned}$$

Collecting the above estimates yields Step 1.

Step 2: There exists two constants C_1 and C_2 such that

$$\|(p + A)^2\varphi\|^2 + \|H_f\varphi\|^2 \leq C_1\|((p + A)^2 + H_f)\varphi\|^2 + C_2\|\varphi\|^2, \quad \forall \varphi \in \mathcal{C} .$$

Calculating a double commutator, we see that

$$\begin{aligned}
&\frac{1}{2}\|H_f\varphi\|^2 + (H_f\varphi, (p + A)^2\varphi) + ((p + A)^2\varphi, H_f\varphi) \\
&= \frac{1}{2}\|H_f\varphi\|^2 + \sum_j 2((p_j + A_j)\varphi, H_f(p_j + A_j)\varphi) \\
&\quad + \sum_j ((\varphi, [A_j, [A_j, H_f]]\varphi) + ([A_j, p_j]\varphi, H_f\varphi) + (H_f\varphi, [A_j, p_j]\varphi)) \\
&\geq \frac{1}{4}\|H_f\varphi\|^2 - C\|(H_f + 1)^{1/2}\varphi\|^2 \\
&\geq -b\|\varphi\|^2,
\end{aligned}$$

for some b . Step 2 follows from this. □

Proof of Theorem 6. By Lemma 11 (b), we know the inclusion $D(p^2 + H_f) \subset D(T_A)$. From the closed graph theorem it follows that T_A is $p^2 + H_f$ bounded. This, Lemma 11 (d), and the fact that \mathcal{C} is an operator core for $p^2 + H_f$, imply that \mathcal{C} is an operator core for T_A . From this, Lemma 11 (a), and Inequality (16) we conclude that $D(T_A) \subset D(p^2 + H_f)$. The statement about the core holds for any closed operators having equal domain. □

Appendix

Proof of Lemma 1. (a). Relation (1) follows from the following relations on \mathcal{F}_{fin} , $[a(f), a(g)] = 0$, $[a^*(f), a^*(g)] = 0$, and $[a(f), a^*(g)] = (f, g)$, for all $f, g \in \mathfrak{h}$.

(b). We will use the natural isomorphism $\otimes^n \mathfrak{h} \cong L^2((\mathbb{R}^d \times \mathbb{C}^p)^n)$. We set $K_n = (k_1, \lambda_1, \dots, k_n, \lambda_n) \in (\mathbb{R}^d \times \mathbb{C}^p)^n$ and write $\int dK_n$ for $\sum_{\lambda_1, \dots, \lambda_n=1}^p \int dk_1 \dots dk_n$. For $\psi \in \mathcal{F}_{\text{fin}}$ and $f \in \mathfrak{h}_\omega$,

$$\begin{aligned} \|a(f)\psi\|^2 &= \sum_{n=0}^{\infty} \int \left| (n+1)^{1/2} \sum_{\lambda_1} \int \bar{f}(k_1) \omega(k_1)^{-1/2} \omega(k_1)^{1/2} \psi_{n+1}(k_1, \lambda_1, K_n) dk_1 \right|^2 dK_n \\ &\leq \|f/\sqrt{\omega}\|^2 \sum_{n=0}^{\infty} \int \sum_{\lambda_1} \int (n+1) \omega(k_1) |\psi_{n+1}(k_1, \lambda_1, K_n)|^2 dk_1 dK_n \\ &= \|f/\sqrt{\omega}\|^2 (\psi, H_f \psi). \end{aligned} \quad (19)$$

Thus $D(H_f^{1/2}) \subset D(a(f))$. For $\varphi \in \mathcal{F}_{\text{fin}}$,

$$\|a^*(f)\varphi\|^2 = (a^*(f)\varphi, a^*(f)\varphi) = \|f\|^2 \|\varphi\|^2 + \|a(f)\varphi\|^2.$$

By this and (19) we find $D(H_f^{1/2}) \subset D(a^*(f))$ and

$$\|a^*(f)\psi\|^2 \leq \|f\|^2 \|\psi\|^2 + \|f/\sqrt{\omega}\|^2 \|H_f^{1/2} \psi\|^2.$$

If $\psi \in \mathcal{F}_{\text{fin}}$ and $f, g \in \mathfrak{h}_\omega$, then with $c_n := (n+1)(n+2)$,

$$\begin{aligned} \|a(f)a(g)\psi\|^2 &= \sum_{n=0}^{\infty} \int \left| c_n^{1/2} \sum_{\lambda_1, \lambda_2} \int \bar{f}(k_1, \lambda_1) \bar{g}(k_2, \lambda_2) \psi_{n+2}(k_1, \lambda_1, k_2, \lambda_2, K_n) dk_1 dk_2 \right|^2 dK_n \\ &\leq \left\| \frac{f}{\sqrt{\omega}} \right\|^2 \left\| \frac{g}{\sqrt{\omega}} \right\|^2 \sum_{n=0}^{\infty} \int \sum_{\lambda_1, \lambda_2} \int c_n \omega(k_1) \omega(k_2) |\psi_{n+2}(k_1, \lambda_1, k_2, \lambda_2, K_n)|^2 dk_1 dk_2 dK_n \\ &\leq \|f/\sqrt{\omega}\|^2 \|g/\sqrt{\omega}\|^2 \|H_f \psi\|^2. \end{aligned} \quad (20)$$

Now using the commutation relations, linearity, and the triangle inequality we can reduce Inequalities (2) and (3) to the estimates (19) and (20).

(c). This follows from the identities $[H_f, a(f)] = -a(\omega f)$ and $[H_f, a^*(f)] = a^*(\omega f)$ on $\mathcal{F}_{\text{fin}} \cap D(H_f)$, which in turn follow from the definition. \square

Proof of Lemma 3. By Lemma 2, the right hand side of (6) is well defined. By definition q is closed if and only if $Q(q)$ is complete under the norm (11). We write (a_n) as a shorthand

notation for the sequence $(a_n)_{n=0}^\infty$. Let $(\varphi_n) \subset Q(q)$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{+1}$, see (11). We see that the sequences

$$(\varphi_n), (H_f^{1/2}\varphi_n), ((p_j + A_j)\varphi_n), j = 1, 2, 3, \quad (21)$$

are Cauchy sequences in \mathcal{H} . Since $(H_f^{1/2}\varphi_n)$ is Cauchy in \mathcal{H} , it follows from Lemma 2 that $(A_j\varphi_n)$ is also Cauchy in \mathcal{H} . Hence also $(p_j\varphi_n)$ is Cauchy in \mathcal{H} . Since p_j and H_f are closed, it follows that the limit $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ is in the domain of $Q(q)$. We conclude $\|\varphi - \varphi_n\|_{+1} \rightarrow 0$ as n tends to infinity. \square

Lemma 13. *Assume $G \in L^\infty(\mathbb{R}^n; \mathfrak{h})$ is weakly ∂_j -differentiable. Then for $\psi \in D(p_j) \cap \mathcal{H}_{\text{fin}}$, $\Phi(G)\psi$ is in the domain of p_j and*

$$p_j\Phi(G)\psi = -i\Phi(\partial_j G)\psi + \Phi(G)p_j\psi.$$

Proof. Suppose

$$\begin{aligned} \psi_1 &= f_1 \otimes \xi_1, & \xi_1 &= a^*(h_1) \cdots a^*(h_M)\Omega, \\ \psi_2 &= f_2 \otimes \xi_2, & \xi_2 &= a^*(g_1) \cdots a^*(g_N)\Omega, \end{aligned}$$

with $f_j \in C_0^\infty(\mathbb{R}^n)$ and $g_i, h_i \in \mathfrak{h}$. Then

$$\begin{aligned} (ip_j\psi_1, \Phi(G)\psi_2) &= (ip_jf_1 \otimes \xi_1, \Phi(G)f_2 \otimes \xi_2) \\ &= \int (\partial_j f_1(x)) [(\xi_1, 2^{-1/2}(a^*(G(x)) + a(G(x))\xi_2) f_2(x)] dx \\ &= 2^{-1/2} \int \left[\partial_j f_1(x) \sum_l (\xi_1, (G(x), g_l) a^*(g_1) \cdots a^*(\widehat{g_l}) \cdots \Omega) f_2(x) \right. \\ &\quad \left. + \sum_l \partial_j f_1(x) ((G(x), h_l) a^*(h_1) \cdots a^*(\widehat{h_l}) \cdots \Omega, \xi_2) f_2(x) \right] dx, \end{aligned}$$

where $a^*(\widehat{h_l})$ stands for the omission of the term $a^*(h_l)$. We find,

$$\begin{aligned} (ip_j\psi_1, \Phi(G)\psi_2) &= -2^{-1/2} \int f_1(x) \left[\sum_l (\xi_1, (g_l, \partial_j G(x)) a^*(g_1) \cdots a^*(\widehat{g_l}) \cdots \Omega) f_2(x) \right. \\ &\quad \left. + \sum_l (\xi_1, (g_l, G(x)) a^*(g_1) \cdots a^*(\widehat{g_l}) \cdots \Omega) \partial_j f_2(x) \right. \\ &\quad \left. + \sum_l ((h_l, \partial_j G(x)) a^*(h_1) \cdots a^*(\widehat{h_l}) \cdots \Omega, \xi_2) f_2(x) \right. \\ &\quad \left. + \sum_l ((h_l, G(x)) a^*(h_1) \cdots a^*(\widehat{h_l}) \cdots \Omega, \xi_2) \partial_j f_2(x) \right] dx \\ &= (\psi_1, -\Phi(\partial_j G)\psi_2) + (\psi_1, \Phi(G)(-ip_j)\psi_2). \end{aligned}$$

Since linear combinations of vectors of the form ψ_1 constitute a core for p_j and p_j is self-adjoint we find $\Phi(G)\psi_2 \in D(p_j)$ and

$$p_j(\Phi(G)\psi_2) = -i\Phi(\partial_j G)\psi_2 + \Phi(G)p_j\psi_2 .$$

This equation now follows for any $\psi_2 \in D(p_j) \cap \mathcal{H}_{\text{fin}}$ by taking linear combinations and then limits. \square

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